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Integral Mean Estimates for Polynomials with Restricted Zeros

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Let $P(z)$ be a polynomial of degree n having all its zeros in $|z| \leqslant K$. For $K=1$, it is known that for each $q>0$,

$$n\left\{\int_0^{2\pi} |P(e^{i\theta})|^q \, d\theta\right\}^{1/q} \leqslant \left\{\int_0^{2\pi} |1+e^{i\theta}|^q \, d\theta\right\}^{1/q} \operatorname*{Max}_{|z|=1} |P'(z)|.$$

In this paper we shall consider the two cases $K \geqslant 1$ and $K<1$, and present certain sharp inequalities.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let $P(z)$ be a polynomial of degree n. It was shown by Turán [9] that if $P(z)$ has all its zeros in $|z| \leqslant 1$, then

$$n \operatorname*{Max}_{|z|=1} |P(z)| \leqslant 2 \operatorname*{Max}_{|z|=1} |P'(z)|. \tag{1}$$

Inequality (1) is best possible with equality for $P(z)=\alpha z^n+\beta$ where $|\alpha|=|\beta|$. As an extension of (1), Govil [3] (see also [1]) proved that if $P(z)$ has all its zeros in $|z| \leqslant K$ where $K \geqslant 1$, then

$$n \operatorname*{Max}_{|z|=1} |P(z)| \leqslant (1+K^n) \operatorname*{Max}_{|z|=1} |P'(z)|. \tag{2}$$

Here equality holds for $P(z)=\alpha z^n+\beta K^n$ where $|\alpha|=|\beta|$ and $K \geqslant 1$. On the other hand, Malik [5] showed that if $P(z)$ has all its zeros in $|z| \leqslant K$ where $K \leqslant 1$, then

$$n \operatorname*{Max}_{|z|=1} |P(z)| \leqslant (1+K) \operatorname*{Max}_{|z|=1} |P'(z)|. \tag{3}$$

Inequality (3) is also best possible with euality for $P(z)=(z+K)^n$.

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Recently Malik [6] obtained a generalization of (1) in the sense that the left-hand side of (1) is replaced by a factor involving the integral mean of $|P(z)|$ on $|z|=1$. In fact, he proved that if $P(z)$ has all its zeros in $|z|\leqslant 1$, then for each $q>0$,

$$n\left\{\int_0^{2\pi}|P(e^{i\theta})|^q\,d\theta\right\}^{1/q}\leqslant\left\{\int_0^{2\pi}|1+e^{i\theta}|^q\,d\theta\right\}^{1/q}\operatorname*{Max}_{|z|=1}|P'(z)|. \tag{4}$$

If we let q tend to infinity in (4), we get (1).

In this paper we shall obtain certain generalizations of the inequalities (2) and (3) which are similar to (4) and thereby present some extensions of (4) also. We prove

THEOREM 1. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z|\leqslant K$ where $K\geqslant 1$, then for each $q\geqslant 1$*

$$n\left\{\int_0^{2\pi}|P(e^{i\theta})|^q\,d\theta\right\}^{1/q}\leqslant\left\{\int_0^{2\pi}|1+K^n\,e^{i\theta}|^q\,d\theta\right\}^{1/q}\operatorname*{Max}_{|z|=1}|P'(z)|. \tag{5}$$

The result is best possible and equality in (5) *holds for the polynomial $P(z)=\alpha z^n+\beta K^n$ where $|\alpha|=|\beta|$.*

Remark 1. Letting $q\to\infty$ in (5), we get (2).

Next we consider the case $K\leqslant 1$ and prove the following

THEOREM 2. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z|\leqslant K$ where $K\leqslant 1$, then for each $q>0$,*

$$n\left\{\int_0^{2\pi}\left|\frac{P(e^{i\theta})}{P'(e^{i\theta})}\right|^q d\theta\right\}^{1/q}\leqslant\left\{\int_0^{2\pi}|1+Ke^{i\theta}|^q\,d\theta\right\}^{1/q}. \tag{6}$$

The result is best possible and equality in (6) *holds for the polynomial $P(z)=(\alpha z+\beta K)^n$ where $|\alpha|=|\beta|$.*

Since $|P'(e^{i\theta})|\leqslant \operatorname{Max}_{|z|=1}|P'(z)|$ for $0\leqslant\theta<2\pi$, the following corollary is an immediate consequence of Theorem 2.

COROLLARY 1. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z|\leqslant K$ where $K\leqslant 1$, then for each $q>0$,*

$$n\left\{\int_0^{2\pi}|P(e^{i\theta})|^q\,d\theta\right\}^{1/q}\leqslant\left\{\int_0^{2\pi}|1+Ke^{i\theta}|^q\,d\theta\right\}^{1/q}\operatorname*{Max}_{|z|=1}|P'(z)|. \tag{7}$$

Remark 2. Letting $q\to\infty$ in (7), we obtain (3). For $K=1$, Corollary 1 reduces to (4).

Finally we present the following interesting generalization of (1) which is also an extension of (4).

THEOREM 3. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant 1$ and $m = \text{Min}_{|z|=1} |P(z)|$, then for every α with $|\alpha| = 1$ and for each $q > 0$,*

$$n \left\{ \int_0^{2\pi} |P(e^{i\theta}) + m\alpha|^q \, d\theta \right\}^{1/q} \leqslant \left\{ \int_0^{2\pi} |1 + e^{i\theta}|^q \, d\theta \right\}^{1/q} \underset{|z|=1}{\text{Max}} |P'(z)|. \quad (8)$$

The result is best possible and equality in (8) holds for the polynomial $P(z) = z^n + \alpha K^n$ where $K \leqslant 1$ and $|\alpha| = 1$.

Letting $q \to \infty$ in (8) and choosing an argument of α suitably, we get the following result.

COROLLARY 2. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leqslant 1$, then*

$$n(\underset{|z|=1}{\text{Max}} |P(z)| + \underset{|z|=1}{\text{Min}} |P(z)|) \leqslant 2 \underset{|z|=1}{\text{Max}} |P'(z)|. \quad (9)$$

The result is best possible and equality in (9) holds for the polynomial $P(z) = z^n + K^n$ where $K \leqslant 1$.

2. PROOFS OF THE THEOREMS

Proof of Theorem 1. Since all the zeros of $P(z)$ lie in $|z| \leqslant K$, $K \geqslant 1$, it follows that the polynomial $G(z) = P(Kz)$ has all its zeros in $|z| \leqslant 1$. Hence the polynomial $H(z) = z^n \overline{G(1/\bar{z})}$ has all its zeros in $|z| \geqslant 1$. Thus, if $z_1, z_2, ..., z_n$ are the zeros of $H(z)$, then $|z_j| \geqslant 1$, $j = 1, 2, ..., n$ and

$$\frac{zH'(z)}{H(z)} = \sum_{j=1}^{n} \frac{z}{z - z_j},$$

so that

$$\text{Re} \frac{e^{i\theta} H'(e^{i\theta})}{H(e^{i\theta})} = \sum_{j=1}^{n} \text{Re} \frac{e^{i\theta}}{e^{i\theta} - z_j} \leqslant \sum_{j=1}^{n} \frac{1}{2} = \frac{n}{2}$$

for points $e^{i\theta}$, $0 \leqslant \theta < 2\pi$, which are not the zeros of $H(z)$. This gives

$$|e^{i\theta} H'(e^{i\theta})/nH(e^{i\theta})| \leqslant |1 - (e^{i\theta} H'(e^{i\theta}))/nH(e^{i\theta})|$$

for points $e^{i\theta}$, $0 \leqslant \theta < 2\pi$, other than the zeros of $H(z)$. Equivalently

$$|H'(e^{i\theta})| \leqslant |nH(e^{i\theta}) - e^{i\theta} H'(e^{i\theta})| \quad (10)$$

for points $e^{i\theta}$, $0 \leqslant \theta < 2\pi$, which are not the zeros of $H(z)$. Since the inequality (10) is trivially true for points $e^{i\theta}$, $0 \leqslant \theta < 2\pi$, which are the zeros of $H(z)$, therefore, it follows that

$$|H'(z)| \leqslant |nH(z) - zH'(z)|, \qquad \text{for} \quad |z| = 1. \tag{11}$$

Since $G(z)$ has all its zeros in $|z| \leqslant 1$, by the Gauss–Lucas theorem, all the zeros of $G'(z)$ also lie in $|z| \leqslant 1$. This implies that the polynomial

$$z^{n-1}\overline{G'(1/\bar{z})} \equiv nH(z) - zH'(z)$$

does not vanish in $|z| < 1$. Therefore, it follows from (11) that the function

$$w(z) = \frac{zH'(z)}{nH(z) - zH'(z)}$$

is analytic for $|z| \leqslant 1$ and $|w(z)| \leqslant 1$ for $|z| = 1$. Furthermore, $w(0) = 0$. Thus the function $1 + w(z)$ is subordinate to the function $1 + z$ for $|z| \leqslant 1$. Hence by a well known property of subordination [4], we have

$$\int_0^{2\pi} |1 + w(e^{i\theta})|^q \, d\theta \leqslant \int_0^{2\pi} |1 + e^{i\theta}|^q \, d\theta, \qquad q > 0. \tag{12}$$

Now

$$1 + w(z) = \frac{nH(z)}{nH(z) - zH'(z)}$$

and

$$|G'(z)| = |z^{n-1}\overline{G'(1/\bar{z})}| = |nH(z) - zH'(z)|, \qquad \text{for} \quad |z| = 1,$$

therefore, for $|z| = 1$,

$$n\,|H(z)| = |1 + w(z)|\,|nH(z) - zH'(z)| = |1 + w(z)|\,|G'(z)|. \tag{13}$$

From (12) and (13) we deduce that for $q > 0$,

$$n^q \int_0^{2\pi} |H(e^{i\theta})|^q \, d\theta \leqslant \int_0^{2\pi} |1 + e^{i\theta}|^q \, d\theta \{\operatorname*{Max}_{|z|=1} |G'(z)|\}^q. \tag{14}$$

If $P(z)$ is a polynomial of degree n which does not vanish in $|z| < 1$, then according to a result of Boas and Rahman [2] we have for every $R \geqslant 1$ and $q \geqslant 1$,

$$\int_0^{2\pi} |P(Re^{i\theta})|^q \, d\theta \leqslant B_q \int_0^{2\pi} |P(e^{i\theta})|^q \, d\theta, \tag{15}$$

where

$$B_q = \int_0^{2\pi} |1 + R^n e^{i\theta}|^q \, d\theta \Big/ \int_0^{2\pi} |1 + e^{i\theta}|^q \, d\theta.$$

Since $H(z)$ does not vanish in $|z| < 1$, we apply (15) with $R = K \geqslant 1$ to $H(z)$ and obtain

$$\int_0^{2\pi} |H(Ke^{i\theta})|^q \, d\theta \leqslant B_q \int_0^{2\pi} |H(e^{i\theta})|^q \, d\theta, \tag{16}$$

where now

$$B_q = \int_0^{2\pi} |1 + K^n e^{i\theta}|^q \, d\theta \Big/ \int_0^{2\pi} |1 + e^{i\theta}|^q \, d\theta \qquad \text{and} \qquad q \geqslant 1.$$

Since

$$H(z) = z^n \overline{G(1/\bar{z})} = z^n \overline{P(K/\bar{z})},$$

it is immediate that for $0 \leqslant \theta < 2\pi$,

$$|H(Ke^{i\theta})| = |K^n e^{in\theta} \overline{P(e^{i\theta})}| = K^n |P(e^{i\theta})|. \tag{17}$$

From (14), (16), and (17) it follows that for $q \geqslant 1$,

$$\begin{aligned} n^q K^{nq} \int_0^{2\pi} |P(e^{i\theta})|^q \, d\theta &\leqslant B_q n^q \int_0^{2\pi} |H(e^{i\theta})|^q \, d\theta \\ &\leqslant \int_0^{2\pi} |1 + K^n e^{i\theta}|^q \, d\theta \{\operatorname*{Max}_{|z|=1} |G'(z)|\}^q. \end{aligned} \tag{18}$$

If $F(z)$ is a polynomial of degree n, then it is a simple deduction from the maximum modulus principle (see [7, Vol. I, p. 137, Problem III, 269] or [8, p. 346]) that

$$\operatorname*{Max}_{|z| = R \geqslant 1} |F(z)| \leqslant R^n \operatorname*{Max}_{|z|=1} |F(z)|. \tag{19}$$

Applying (19) to the polynomial $G'(z) = KP'(Kz)$, which is of degree $n-1$, we get

$$\operatorname*{Max}_{|z|=1} |G'(z)| = K \operatorname*{Max}_{|z|=1} |P'(Kz)| = K \operatorname*{Max}_{|z| = K \geqslant 1} |P'(z)| \leqslant K^n \operatorname*{Max}_{|z|=1} |P'(z)|. \tag{20}$$

Using (20) in (18), we finally obtain

$$n^q \int_0^{2\pi} |P(e^{i\theta})|^q \, d\theta \leqslant \int_0^{2\pi} |1 + K^n e^{i\theta}|^q \, d\theta \, \{\operatorname*{Max}_{|z|=1} |P'(z)|\}^q,$$

which is equivalent to (5) and this completes the proof of Theorem 1.

Proof of Theorem 2. Since the polynomial $P(z)$ has all its zeros in $|z| \leqslant K \leqslant 1$, it follows that the polynomial $G(z) = P(Kz)$ has all its zeros in $|z| \leqslant 1$. Therefore, if $z_1, z_2, ..., z_n$ are the zeros of $G(z)$, then $|z_j| \leqslant 1$, $j = 1, 2, ..., n$, and

$$\frac{zG'(z)}{G(z)} = \sum_{j=1}^{n} \frac{z}{z - z_j},$$

so that for points $e^{i\theta}$, $0 \leqslant \theta < 2\pi$, other than the zeros of $G(z)$ we have

$$\operatorname{Re} \frac{e^{i\theta} G'(e^{i\theta})}{G(e^{i\theta})} = \sum_{j=1}^{n} \operatorname{Re} \frac{e^{i\theta}}{e^{i\theta} - z_j} \geqslant \sum_{j=1}^{n} \frac{1}{2} = \frac{n}{2}.$$

This gives by similar reasoning as in the proof of Theorem 1 that

$$|nG(z) - zG'(z)| \leqslant |G'(z)|, \qquad \text{for} \quad |z| = 1. \tag{21}$$

Since by the Gauss–Lucas theorem the polynomial $G'(z)$ has all its zeros in $|z| \leqslant 1$, by the maximum modulus principle it follows that the inequality (21) holds for $|z| > 1$ also. Replacing $G(z)$ by $P(Kz)$ and $G'(z)$ by $KP'(Kz)$ in (21), we obtain

$$|nP(Kz) - KzP'(Kz)| \leqslant K\,|P'(Kz)|, \qquad \text{for} \quad |z| \geqslant 1. \tag{22}$$

Since $K \leqslant 1$, we take in particular $z = e^{i\theta}/K$, $0 \leqslant \theta < 2\pi$, in (22) to get

$$|nP(e^{i\theta}) - e^{i\theta} P'(e^{i\theta})| \leqslant K\,|P'(e^{i\theta})|.$$

This shows that

$$|nP(z) - zP'(z)| \leqslant K|P'(z)|, \qquad \text{for} \quad |z| = 1. \tag{23}$$

If $Q(z) = z^n \overline{P(1/\bar{z})}$, then $P(z) = z^n \overline{Q(1/\bar{z})}$ and it can b easily seen that for $|z| = 1$,

$$|Q'(z)| = |nP(z) - zP'(z)| \qquad \text{and} \qquad |P'(z)| = |nQ(z) - zQ'(z)|. \tag{24}$$

Using (24) in (23), we get

$$|Q'(z)| \leqslant K\,|nQ(z) - zQ'(z)|, \qquad \text{for} \quad |z| = 1. \tag{25}$$

Since $P(z)$ has all its zeros in $|z| \leqslant K \leqslant 1$, it follows by the Gauss–Lucas theorem that all the zeros of $P'(z)$ also lie in $|z| \leqslant K \leqslant 1$. This shows that the polynomial

$$z^{n-1} \overline{P'(1/\bar{z})} \equiv nQ(z) - zQ'(z)$$

has all its zeros in $|z| \geqslant (1/K) \geqslant 1$. Therefore, it follows from (25) that the function

$$w(z) = \frac{zQ'(z)}{K(nQ(z) - zQ'(z))}$$

is analytic in $|z| \leqslant 1$ and $|w(z)| \leqslant 1$ for $|z| = 1$. Also $w(0) = 0$. Thus the function $1 + Kw(z)$ is subordinate to the function $1 + Kz$ for $|z| \leqslant 1$. Hence by a well known property of subordination [4], we have for each $q > 0$,

$$\int_0^{2\pi} |1 + Kw(e^{i\theta})|^q \, d\theta \leqslant \int_0^{2\pi} |1 + Ke^{i\theta}|^q \, d\theta. \tag{26}$$

Now

$$1 + Kw(z) = \frac{nQ(z)}{nQ(z) - zQ'(z)},$$

which gives with the help of (24) that for $|z| = 1$,

$$n|Q(z)| = |1 + Kw(z)| \, |nQ(z) - zQ'(z)| = |1 + Kw(z)| \, |P'(z)|. \tag{27}$$

Since $|P(z)| = |Q(z)|$ for $|z| = 1$, therefore, from (27) we get

$$n|P(z)| = |1 + Kw(z)| \, |P'(z)|, \qquad \text{for} \quad |z| = 1. \tag{28}$$

From (26) and (28) we deduce that for each $q > 0$,

$$n^q \int_0^{2\pi} |P(e^{i\theta})/P'(e^{i\theta})|^q \, d\theta \leqslant \int_0^{2\pi} |1 + Ke^{i\theta}|^q \, d\theta,$$

which is equivalent to (6) and this completes the proof of Theorem 2.

Proof of Theorem 3. If $P(z)$ has a zero on $|z| = 1$, then $m = \operatorname{Min}_{|z|=1} |P(z)| = 0$ and the result follows from Corollary 1 by taking $K = 1$. Henceforth we suppose that all the zeros of $P(z)$ lie in $|z| < 1$. If $Q(z) = z^n \overline{P(1/\bar{z})}$, then $\operatorname{Min}_{|z|=1} |Q(z)| = \operatorname{Min}_{|z|=1} |P(z)| = m > 0$. Since all the zeros of $Q(z)$ lie in $|z| > 1$ and $m \leqslant |Q(z)|$ for $|z| = 1$, therefore, by the maximum modulus principle it follows that $m \leqslant |Q(z)|$ for $|z| \leqslant 1$. Replacing z by $1/\bar{z}$ and noting that $z^n \overline{Q(1/\bar{z})} = P(z)$, we conclude that

$$m|z|^n \leqslant |P(z)|, \qquad \text{for} \quad |z| \geqslant 1. \tag{29}$$

Now consider the polynomial

$$G(z) = P(z) + \alpha m,$$

where α is a complex number such that $|\alpha| = 1$. Then all the zeros of $G(z)$ lie in $|z| \leqslant 1$. Because if for some $z = z_0$, with $|z_0| > 1$,

$$G(z_0) = P(z_0) + \alpha m = 0,$$

then we have $|P(z_0)| = |\alpha m| = m < m|z_0|^n$. But this is a contradiction to (29). Thus for every α, with $|\alpha| = 1$, the polynomial $G(z) = P(z) + \alpha m$ has all its zeros in $|z| \leqslant 1$. Applying Corollary 1 with $K = 1$ to the polynomial $G(z)$ and noting that $G'(z) = P'(z)$, we get for each $q > 0$,

$$n\left\{\int_0^{2\pi} |P(e^{i\theta}) + \alpha m|^q \, d\theta\right\}^{1/q} \leqslant \left\{\int_0^{2\pi} |1 + e^{i\theta}|^q \, d\theta\right\}^{1/q} \operatorname*{Max}_{|z|=1} |P'(z)|,$$

which is the conclusion of Theorem 3.

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